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МАТЕМАТИКА 3

• решени рокови •



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Ispit iz Matematike 3

Zadaci

1. (15) Izračunati zapreminu tela ograničenog površinom

$$(x^2 + y^2 + z^2)^2 = az(x^2 + y^2), \quad (a > 0).$$

2. Date su površi $S_1 : z = 4 - x^2 - y^2$ i $S_2 : x^2 + y^2 = 2x$.

1° (10) Primenom STOKESove teoreme izračunati $\oint_C y \, dx + z \, dy + x \, dz$, gde kriva C nastaje presekom površi S_1 i S_2 pozitivno orijentisana ako se posmatra iz tačke $(3, 0, 0)$.

- 2° (10) Izračunati

$$\iint_S \left(\frac{x^3}{3} - xy^2 \right) \, dy \, dz + \left(\frac{y^3}{3} - x^2y \right) \, dz \, dx + \left(\frac{z^2}{2} - xy \right) \, dx \, dy,$$

gde je S spoljna strana tela ograničenog površima S_1 i S_2 (unutar površi S_2) i ravni $z = 0$.

Odseci OE, OF i OS - N. Cakić

3. (20) Kompleksnom integracijom izračunati $\int_0^{+\infty} \frac{dx}{(x^2 + 1)\sqrt[3]{x}}$.

4. 1° (8) Koristeći teoremu kašnjenja odrediti $L\{f(t)\}$ ako je $f(t) = \begin{cases} t, & 0 < t < 1 \\ e^{-(t-1)}, & t > 1 \end{cases}$

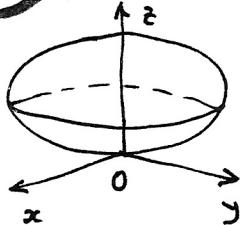
- 2° (7) Naći $L^{-1}\{F(s)\}$ gde je $F(S) = \frac{se^{-s}}{s^2 + 2s + 2}$.

Odseci OT i OG - S. Ješić

3. (20) Funkciju $f(x) = \frac{a \sin x}{1 - 2a \cos x + a^2}$, $|a| > 1$, predstaviti FOURIERovim redom na intervalu $[-\pi, \pi]$.

4. (15) Odrediti LAPLACEovu transformaciju $F(p)$ funkcije $f(t) = \int_0^{+\infty} \frac{\cos tx}{x^2 + a^2} \, dx$, za $a > 0$, a zatim pokazati da je $f(t) = \frac{\pi}{2a} e^{-at}$.

$$1. (x^2 + y^2 + z^2)^2 = a_z(x^2 + y^2) \geq 0 \Rightarrow z \geq 0 \text{ Повыше изнайд рзвни } z=0$$



Уведимо сферне координате. Једначина повези ге

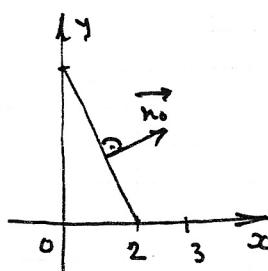
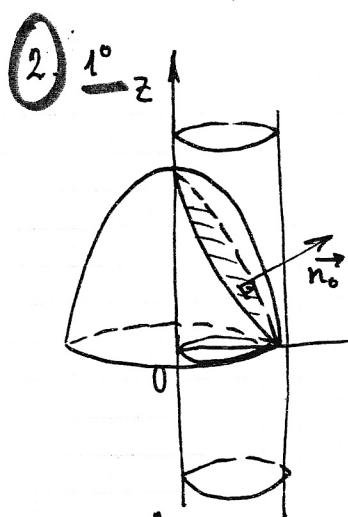
$$P = a \sin^2 \theta \cos^2 \theta \Rightarrow \sin \theta \geq 0 \Rightarrow \theta \in [0, \pi/2]$$

$$\varphi \in [0, 2\pi], \quad 0 \leq s \leq a \sin \theta \cos^2 \theta$$

$$V = \iiint_S s^2 \cos^2 \theta \, ds \, d\varphi \, d\theta = \frac{2\pi}{3} a^3 \int_0^{\pi/2} \sin^3 \theta \cos^2 \theta \, d\theta = \frac{2\pi}{3} a^3 \int_0^{\pi/2} (1 - \cos^2 \theta) \cos^2 \theta \sin \theta \, d\theta$$

Ако у последњи интеграл уведемо смислу $\cos\theta = t$

$$\text{ДОБИЈАМО} \quad V = \frac{a^3 \pi}{60}$$



Повърхни S_1 и S_2 сечу се по краи b от C
која прилага равни:

$$\pi : z = 4 - 2x$$

Кривъ съ равни отрезъци по гръден S.

Примитив Stokes-ове теореме и мадж

$$I = \oint_C \int \int_S \left| \begin{array}{cccc} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{array} \right| dS = - \int \int_S (\cos \alpha + \cos \beta + \cos \gamma) dS$$

Now to see if $z = 4 - 2x$, $\bar{z}_x = -2$, $\bar{z}_y = 0$ $y = f(\bar{z}_0, \bar{x})$ will do this

$$\text{To find } \cos \theta = \frac{1}{\sqrt{1+2^2+x_2^2}} = \frac{1}{\sqrt{5}}, \cos \alpha = \frac{2}{\sqrt{5}}, \cos \beta = 0, \text{ and } \theta$$

$$I = - \iint_S \left(\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} \right) dS = - \frac{3}{\sqrt{5}} \iint_S dS = - \frac{3}{\sqrt{5}} \iint_{x^2+y^2 \leq 2} \sqrt{1+z_x^2+z_y^2} dx dy$$

$$= -3 \iint_{x^2+y^2 \leq 2x} dx dy = -3\pi$$

2° Применяя формулы $\sigma_{\text{струн}}$ и $\sigma_{\text{макс}}$

$$I = \iiint_V x^2 dx dy dz \quad \text{в цилиндрических координатах}$$

$$I = \iiint_{-\pi/2 \leq \varphi \leq \pi/2} g \cdot z \, dz \, ds \, d\varphi = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2 \cos \varphi} s \, ds \int_0^{4-s^2} z \, dz = \frac{1}{2} \int_{-\pi/2}^{\pi/2} d\varphi \left(-\frac{1}{6} (4-s^2)^3 \right) \Big|_0.$$

$$= \frac{64}{12} \int_{-\pi/2}^{\pi/2} (1 - \sin^6 \varphi) d\varphi = \frac{128}{12} \int_0^{\pi/2} (1 - \sin^6 \varphi) d\varphi$$

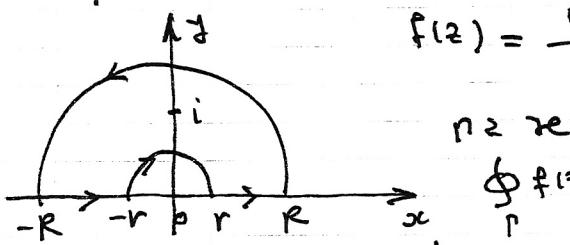
$$= \frac{128}{3} \left[\frac{\pi}{2} - \int_0^{\pi/2} \sin^6 p \, dp \right] = \frac{32}{3} \left(\frac{\pi}{2} - \frac{5!!}{6!!} \frac{\pi}{2} \right) = \frac{11}{3} \pi$$

о ф, о с , о р

$$③ I = \int_0^{+\infty} \frac{x^{-1/3}}{x^2+1} dx = \int_0^{+\infty} \frac{e^{-\frac{1}{3} \ln x}}{x^2+1} dx, \text{ при дружино компонент интеграл}$$

$\oint_{\Gamma} \frac{e^{-\frac{1}{3} \ln z}}{z^2+1} dz$, где зе Γ контурд на слици.

$$f(z) = \frac{e^{-\frac{1}{3} \ln z}}{z^2+1} \text{ само } z_1 \geq i \text{ (пол првог реда) припада је } \Gamma$$



$$\oint_{\Gamma} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z)$$

$$\operatorname{Res}_{z=i} f(z) = \operatorname{Res}_{z=i} \frac{e^{-\frac{1}{3} \ln z}}{(z^2+1)'} = \left. \frac{e^{-\frac{1}{3} \ln z}}{2z} \right|_{z=i} = \frac{e^{-\frac{1}{3} \ln i}}{2i} = \frac{e^{-i\pi/6}}{2i}$$

$$\text{Па зе } \oint_{\Gamma} f(z) dz = 2\pi i \cdot \frac{e^{-i\pi/6}}{2i}, \text{ односно}$$

$$\int_{-r}^R + \int_{C_R} + \int_{-R}^r + \int_{C_r} = R e^{-i\pi/6} \quad (1)$$

$$\left| \int_{C_R} f(z) dz \right| \leq \int_0^\pi \frac{e^{-\frac{1}{3} \ln R} \cdot |e^{-\frac{1}{3} i\varphi}|}{|Re^{i\varphi}| - 1} R |e^{i\varphi}| d\varphi = \frac{\sqrt[3]{R^2}}{R^2 - 1} \pi \rightarrow 0 \quad R \rightarrow \infty \quad (2)$$

$$\left| \int_{C_r} f(z) dz \right| \leq \int_0^\pi \frac{e^{-\frac{1}{3} \ln r} |e^{-\frac{1}{3} i\varphi}|}{|r^2 e^{i2\varphi} + 1|} r |e^{i\varphi}| d\varphi \leq \int_0^\pi \frac{r^{-1/3}}{1 - r^2} r d\varphi = \frac{\sqrt[3]{r^2}}{1 - r^2} \pi \rightarrow 0 \quad r \rightarrow 0 \quad (3)$$

$$\int_{-r}^{-R} \frac{e^{-\frac{1}{3} \ln x} e^{ix}}{(x e^{i\pi})^2 + 1} e^{ix} dx = \int_{-r}^R \frac{e^{-\frac{1}{3} \ln x} e^{-i\pi/3}}{x^2 + 1} dx \quad (4)$$

Ако у је, пустимо да $r \rightarrow \infty$, $r \rightarrow 0$ и испоредимо резултате (2), (3) и (4)

Добијамо

$$(1 + e^{-i\pi/3}) I = \pi e^{-i\pi/6} \Rightarrow I = \pi \frac{e^{-i\pi/6}}{1 + e^{-i\pi/3}} = \pi \frac{e^{-i\pi/6}}{e^{-i\pi/6}(e^{i\pi/6} + e^{-i\pi/6})} = \frac{\pi}{2 \cos \pi/6} = \frac{\pi}{\sqrt{3}}$$

4. 10

$$f(t) = t(h(t) - h(t-1)) + e^{-(t-1)} h(t-1)$$

$$= t h(t) - [1 + h(t-1)] e^{-(t-1)} h(t-1)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} \left[\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s+1} \right] e^{-s}$$

$$2^{\circ} \quad \mathcal{L}^{-1}\{F(s)\} = f(t-1) h(t-1) \quad \text{где зе}$$

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 2s + 2} \right\} = \mathcal{L}^{-1}\left\{ \frac{(s+1) - 1}{(s+1)^2 + 1} \right\} = \mathcal{L}^{-1}\left\{ \frac{s+1}{(s+1)^2 + 1} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2 + 1} \right\}$$

$$= e^{-t} \cos t - e^{-t} \sin t = e^{-t} (\cos t - \sin t)$$

$$\mathcal{L}^{-1}\{f(t)\} = e^{-(t-1)} \Gamma_{n+1} (t-1) - \dots - e^{-(t-n)} \Gamma_{n+1} (t-n).$$

Ispit iz Matematike 3

Zadaci

1. (13) Odrediti ekstremne vrednosti funkcije $f(x, y) = x^2ye^{2x-y}$.

2. Dat je eliptički paraboloid $3x^2 + y^2 = 3 - z$.

1° (10) Izračunati površinski integral

$$I_1 = \iint_S (2xy - x^2) dy dz + (2yz - y^2) dz dx + (2zx - 2z^2) dx dy,$$

gde je S spoljna strana tela ograničenog datim paraboloidom i ravni $z = 0$.

2° (4+8) Direktnom metodom izračunati krivolonijski integral

$$I_2 = \oint_C y dx + z dy + x dz,$$

gde je kriva C presek datog paraboloida i paraboličkog cilindra $z = 2y^2$ pozitivno orijentisana ako se posmatra iz tačke $(0, 0, 3)$. Rezultat proveriti primenom STOKESove formule.

Odseci OE, OF i OS - N. Cakić

3. (20) Kompleksnom integracijom izračunati $\int_0^{+\infty} \frac{\ln x}{x^4 + 1} dx$.

4. (15) Primenom \mathcal{L} -transformacije odrediti partikularno rešenje sistema diferencijalnih jednačina

$$x'(t) + y'(t) - y(t) = 0$$

$$x'(t) + x(t) + 2y'(t) = f(t)$$

koje zadovoljava početne uslove $x(0) = y(0) = 0$, gde je

$$f(t) = \begin{cases} 0, & t < 1 \\ E_0 \cos \omega(t-1), & t \geq 1 \end{cases}, \quad \omega \in \mathbb{R}, \omega \neq 1.$$

Odseci OT i OG - S. Ješić

3. (20) Kompleksnom integracijom izračunati $\int_0^{+\infty} \frac{x^2 - a^2}{x^2 + a^2} \frac{\sin x}{x} dx, \quad (a > 0)$.

4.(15) Funkciju $f(x) = \max(\sin x, 0)$ predstaviti FOURIEROVIM redom na intervalu $[-\pi, \pi]$, a zatim na osnovu dobijenog rezultata izračunati sumu brojnog

$$\text{reda } \sum_{n=1}^{+\infty} \frac{1}{4n^2 - 1}.$$

$$f(x, y) = xy^2 e^{x-2y}$$

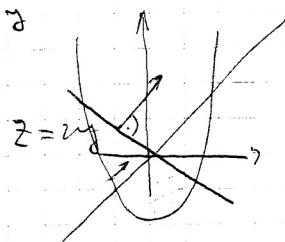
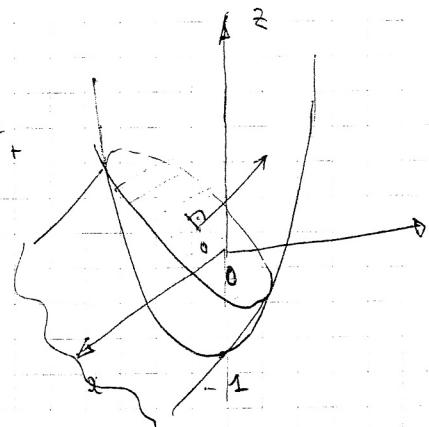
$$2x^2 + y^2 - \frac{z^2}{4} = 1$$

$$z = 2x^2 + y^2 - 1$$

$$z = 2y$$

$$z = 2x^2 + y^2$$

$$z^2 = 2x^2 +$$



ss

1

$$2x^2 + y^2 = 2y$$

$$2x^2 + (y-1)^2 = 1$$

$$f(x, y) = x^2 y e^{2x-y}$$

$$f'_x = 2xy e^{2x-y} + x^2 y e^{2x-y} \cdot 2$$

$$f'_x = (2xy + 2x^2 y) e^{2x-y}$$

$$f'_x = 2xy(1+x) e^{2x-y} = 0$$

$$f'_y = x^2 e^{2x-y} + x^2 y e^{2x-y} \cdot (-1)$$

$$f'_y = x^2(1+y) e^{2x-y} = 0$$

$x=0$ $M_1(0, y)$ критич. точка

$x \neq 0$ $M_2(-1, 1)$

$$f''_{xx} = y \left[(2+4x) e^{2x-y} + (2x+2y^2) 2e^{2x-y} \right]$$

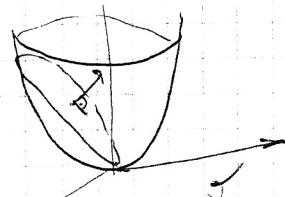
$$f''_{xx} = y(2+8x+4x^2) e^{2x-y}$$

$$f''_{xy} = 2x(1+x)(1-y) e^{2x-y}$$

$$f''_{yy} = x^2 \left[-e^{2x-y} + (1-y) e^{2x-y} \cdot (-1) \right]$$

$$f''_{yy} = x^2 [(y-2) e^{2x-y}]$$

$$z = 2x^2 + y^2$$



$M_1(0, y)$

$$B = 0$$

$$C = 0 \quad AC - B^2 > 0$$

условия неизм. макс

$M_2(-1, 1)$

$$A = (2-8+4) e^{-3} = -2e^{-3} < 0$$

$$B = 0$$

$$C = -e^{-3} \quad AC - B^2 = 2e^{-6} > 0$$

макс

$$f_{\max}(-1, 1) = e^{-3}$$

$$\Delta f(0, \Delta y) = f(0, 1, \Delta y) - f(0, 1, 0)$$

$$\Delta f(0, \Delta y) = \Delta x^2 (1+\Delta y) e^{2x-1(\Delta y)}$$

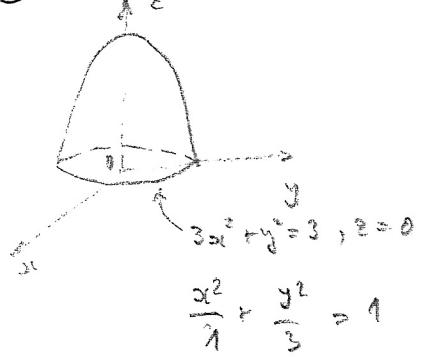
$y > 0 \quad \Delta f(0, \Delta y) \geq 0$ несущий

$y < 0 \quad \Delta f(0, \Delta y) \leq 0$ несущий

$$(0, 0) \quad \Delta f(0, 0) = \Delta x \Delta y e^{2x-0y} \geq 0$$

однозначно
единственна.

$$② 3x^2 + y^2 = 3 - z$$

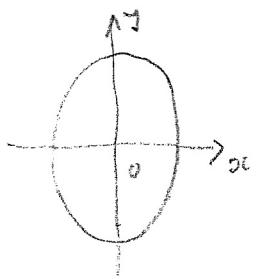


$$I_1 = \iiint_V (2y - 2z + 2z - \cancel{y^2} + \cancel{2z} - 4z) dx dy dz$$

$$= -2 \iiint_V z dx dy dz$$

уравнение симметрии вдоль оси z

$$\begin{cases} x = s \cos \varphi \\ y = \sqrt{3} s \sin \varphi \\ z = z \end{cases} \quad J = \sqrt{3} s$$



$$0 \leq z \leq 3 - 3s^2 - y^2$$

$$0 \leq z \leq 3 - 3s^2 \cos^2 \varphi - 3s^2 \sin^2 \varphi$$

$$0 \leq z \leq 3 - 3s^2, \quad 0 \leq s \leq 1, \quad 0 \leq \varphi \leq 2\pi$$

$$I_1 = -2 \iiint_{0 \leq \varphi \leq 2\pi} z \sqrt{3} s \, ds \, d\varphi \, dz$$

$$0 \leq s \leq 1$$

$$0 \leq z \leq 3(1-s^2)$$

$$= -2\sqrt{3} \int_0^{2\pi} d\varphi \int_0^1 s \, ds \int_0^{3(1-s^2)} z \, dz$$

$$= -2\sqrt{3} \cdot 2\pi \int_0^1 s \, ds \frac{1}{2} [3(1-s^2)]^2$$

$$= -18\sqrt{3}\pi \int_0^1 (1-s^2)^2 s \, ds = -9\sqrt{3}\pi \int_0^1 (1-s^2)^2 2s \, ds$$

$$1-s^2=t$$

$$-2s \, ds = dt$$

$$\text{сумма} \quad = -9\sqrt{3}\pi \int_1^0 t^2 \cdot (-dt)$$

$$= -9\sqrt{3}\pi \int_0^1 t^2 dt = -9\sqrt{3}\pi \frac{1}{3}t^3 \Big|_0^1 = \boxed{-3\sqrt{3}\pi}$$

2. Определите массу фигуры C в координатной системе Oxy если фигура

$$3x^2 + y^2 = 5 - 2y^2 \text{ при } x^2 + y^2 = 1$$

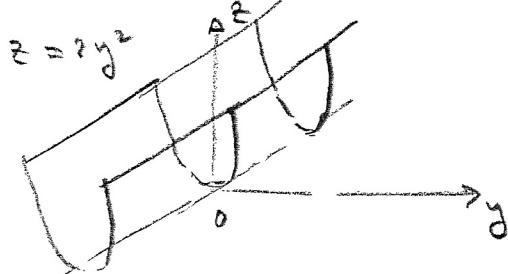
Параллельное сечение фигуры C плоскостью

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = 2 \sin^2 t = 1 - \cos 2t \end{cases} \quad 0 \leq t \leq 2\pi$$

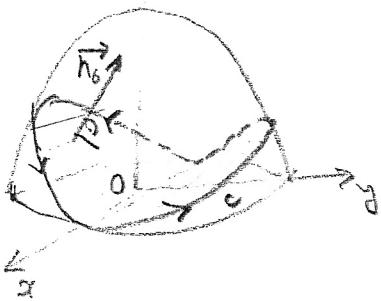
$$I_2 = \int_0^{2\pi} [\sin t \cdot (-\sin t) + (1 - \cos 2t) \cos t + \cos t \cdot (-2 \sin t)] dz$$

= π (Все члены имеют одинаковые коэффициенты, поэтому $I_1 + I_2 = 0$)

Stokes' theorem



S je loci избраних координатног прстенца $z = 2y^2$ са основом на кружу с ограниченим прстеном нормале



$$I_2 = \iint_S \left| \begin{matrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{matrix} \right| dS$$

$$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$$

$$0 < \delta < \pi/2$$

$$= - \iint_S (\cos\alpha + \cos\beta + \cos\gamma) dS$$

ограниченка координата избрана S на раван Oxy тј $\sqrt{x^2 + y^2} \leq 1$

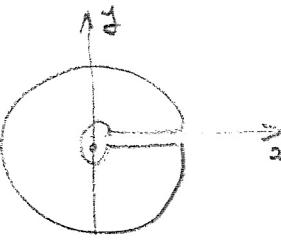
$$S: z = 2y^2 \Rightarrow z_x = 0, z_y = 4y$$

$$\cos\alpha = \frac{1}{\sqrt{1+2y^2+2y^2}} = \frac{1}{\sqrt{1+16y^2}}, \cos\beta = 0, \cos\gamma = \frac{-4y}{\sqrt{1+16y^2}}$$

$$I_2 = \iint_S \left(\frac{-4y + 1}{\sqrt{1+16y^2}} \right) dS = - \iint_{x^2+y^2 \leq 1} \frac{-4y + 1}{\sqrt{1+16y^2}} \sqrt{1+16y^2} dx dy$$

$$= \iint_{x^2+y^2 \leq 1} y dx dy - \iint_{x^2+y^2 \leq 1} dx dy = -\pi$$

$$\int_0^{\infty} \frac{\ln x}{(x+1)(x^2+1)} dx$$



$$\oint_C \frac{\ln z}{(z+1)(z^2+1)} dz$$

$$f(z) = \frac{\ln z}{(z+1)(z^2+1)}$$

$$(3) \int_0^{+\infty} \frac{\ln x}{x^2+1} dx$$

$$\oint_C \frac{\ln z}{z^2+1} dz$$

$$f(z) = \frac{\ln z}{z^2+1} \quad z^4 = -1 \quad z^4 = e^{i\pi} \quad z = e^{i\frac{(2k+1)\pi}{4}} \quad k=0,1,2,3$$

$$z_0 = e^{i\pi/4}, \quad z_1 = e^{i3\pi/4} \in \text{int } P$$

$$z_2 = e^{i5\pi/4}, \quad z_3 = e^{i7\pi/4} \notin \text{int } P$$

$$z=0 \notin \text{int } P$$

$$\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4}$$

$$e^{i\pi/4} = e^{i(\pi + \pi/4)} \\ = e^{i5\pi/4}$$

$$\int_r^R + \int_{S_R} + \int_{-r}^{-R} + \int_{C_R} = 2\pi i \sum \operatorname{Res} f(z)$$

$$\operatorname{Res} f(z) = \left. \frac{\ln z}{4z^3} \right|_{z=e^{i\pi/4}} = \frac{i\pi/4}{4e^{i3\pi/4}} = i\frac{\pi}{16} e^{-i\frac{3\pi}{4}} = i\frac{\pi}{16} \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right)$$

$$\operatorname{Res} f(z) = \left. \frac{\ln z}{4z^3} \right|_{z=e^{i3\pi/4}} = \frac{i3\pi/4}{-4e^{i9\pi/4}} = i\frac{3\pi}{16} e^{i\pi/4} = i\frac{3\pi}{16} \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \right)$$

$$\sum \operatorname{Res} f(z) = i\frac{\pi}{16} \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} + 3\frac{\sqrt{2}}{2} - 3i\frac{\sqrt{2}}{2} \right) = i\frac{\pi}{16} \left(2 \cdot \frac{\sqrt{2}}{2} - 4i\frac{\sqrt{2}}{2} \right)$$

$$= i\frac{\pi}{16} (\sqrt{2} - 2i\sqrt{2}) = i\frac{\sqrt{2}}{16} + \frac{5\sqrt{2}}{8}$$

$$\int_0^{+\infty} + \int_{-\infty}^0 \frac{\ln x e^{it}}{x^2+1} e^{it} dx = 2\pi i \left(i\frac{\pi\sqrt{2}}{16} + \frac{5\sqrt{2}}{8} \right)$$

$$\int_0^{+\infty} + \int_0^{+\infty} \frac{w\alpha + \bar{r}}{x^4 + 1} d\alpha = 2\pi i \left(i \frac{\pi \sqrt{2}}{16} + \frac{\pi \sqrt{2}}{8} \right)$$

$$2 \int_0^{+\infty} \frac{w\alpha}{x^4 + 1} d\alpha + i\pi \int_0^{+\infty} \frac{d\alpha}{x^4 + 1} = -2 \frac{\pi^2 \sqrt{2}}{16} + 2\pi i \frac{\pi \sqrt{2}}{8}$$

$$\int_0^{+\infty} \frac{w\alpha}{x^4 + 1} d\alpha = -\frac{\pi^2 \sqrt{2}}{16}, \quad \int_0^{+\infty} \frac{d\alpha}{x^4 + 1} = \frac{\pi \sqrt{2}}{4}$$

$$\int_0^{+\infty} \frac{w^2 z}{z^4 + 1} dz, \quad f(z) = \frac{w^2 z}{z^4 + 1}$$

$$\sum \text{Res}(f(z)) = -\frac{\pi^2}{64} \left(-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) - \frac{9\pi^2}{16} \left(\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right)$$

$$\text{Res}_{z=i\sqrt{2}/4} f(z) = \frac{w^2 z}{4z^3} \Big|_{z=i\sqrt{2}/4} = \frac{(i\sqrt{2}/4)^2}{4e^{i3\pi/4}} = -\frac{\pi^2}{64} e^{-i3\pi/4}$$

$$\text{Res}_{z=0} f(z) = \frac{w^2 z}{4z^3} \Big|_{z=0} = \frac{(i3\pi/4)^2}{4e^{i3\pi/4}} = -\frac{9\pi^2}{64} e^{-i\pi/4}$$

$$\sum \text{Res}(f(z)) = -\frac{\pi^2}{64} \left[-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} + 9 \frac{\sqrt{2}}{2} - 9i \frac{\sqrt{2}}{2} \right]$$

$$= -\frac{9\pi^2}{64} \left[8 \frac{\sqrt{2}}{2} - 10i \frac{\sqrt{2}}{2} \right] = -\frac{\sqrt{2}}{16} \pi^2 + i \frac{5\sqrt{2}}{64} \pi^2$$

$$\int_0^{+\infty} + \int_{-\infty}^0 \frac{(w\alpha + i\pi)^2}{x^4 + 1} (-1) d\alpha = 2\pi i \left(-\frac{\sqrt{2}}{16} \pi^2 + i \frac{5\sqrt{2}}{64} \pi^2 \right)$$

$$2 \int_0^{+\infty} \frac{w^2 z}{z^4 + 1} dz \pi^2 + 2i\pi \int_0^{+\infty} \frac{w\alpha}{x^4 + 1} d\alpha = -i \frac{\sqrt{2}}{8} \pi^3 - \frac{5\sqrt{2}}{32} \pi^3$$

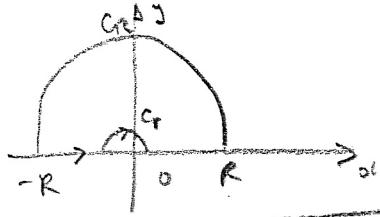
$$\int_0^{+\infty} \frac{w\alpha}{x^4 + 1} d\alpha = -\frac{\sqrt{2}}{16} \pi^2$$

$$2 \int_0^{+\infty} \frac{w^2 z}{z^4 + 1} dz - \pi^2 \cdot \frac{\sqrt{2}}{4} = -\frac{5\sqrt{2}}{32} \pi^3$$

$$2 \int_0^{+\infty} \frac{w^2 z}{z^4 + 1} dz = \frac{\sqrt{2}}{4} \pi^3 - \frac{5\sqrt{2}}{32} \pi^3$$

$$\int_0^{+\infty} \frac{w^2 z}{z^4 + 1} dz = \frac{3\sqrt{2}}{64} \pi^3$$





$$|zf(z)| \leq |z| \frac{\sqrt{m|z|^2 + 4r^2}}{|z|^4 - 1} \leq |z| \frac{\sqrt{m|z|^2 + 4r^2}}{|z|^4 - 1}$$

$$z \in C_r \quad |zf(z)| \leq \frac{R}{R^4 - 1} \sqrt{mR + 4r^2} \rightarrow 0 \quad R \rightarrow \infty \quad |z| = R \rightarrow \infty$$

так же $\lim_{r \rightarrow \infty} \int_{\gamma_r} f(z) dz = 0$ по теореме Коши

$$z \in \gamma_r \quad |zf(z)| \leq \frac{r \sqrt{m^2 r + 4r^2}}{1 - r^2} \rightarrow 0 \quad r \rightarrow \infty$$

$\lim_{r \rightarrow 0} \int_{\gamma_r} f(z) dz = 0$ по теореме Коши

$$\begin{aligned} & \left(\frac{i\pi}{16} \left(-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) + i \frac{3\pi}{16} \left(\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) \right) \\ &= i \frac{\pi}{16} \left[-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} + 3 \frac{\sqrt{2}}{2} - 3i \frac{\sqrt{2}}{2} \right] = i \frac{\pi}{16} (\sqrt{2} - 2i\sqrt{2}) = i \frac{\pi}{16} \sqrt{2} + i \frac{\pi}{8} \sqrt{2} \end{aligned}$$

$$(4) \quad x'(t) + y'(t) - y(t) = 0$$

$$x'(t) + x(t) + 2y'(t) = f(t)$$

$$f(t) = E_0 \cos \omega(t-1) \cdot h(t-1)$$

$$\Im X(s) + \Im Y(s) - Y(s) = 0$$

$$\Im' X(s) + X(s) + 2\Im Y(s) = F(s)$$

$$F(s) = E_0 e^{-s} \frac{s}{s^2 + \omega^2}$$

$$\Im X(s) + (s-1)Y(s) = 0$$

$$(s+1)X(s) + 2sY(s) = E_0 \frac{s}{s^2 + \omega^2} e^{-s}$$

$$2s^2 - (s^2 - 1) = s^2 + 1$$

$$D = \begin{vmatrix} s & s-1 \\ s+1 & 2s \end{vmatrix} = 2s^2 - s^2 + 1 = s^2 + 1$$

$$\Delta_1 = \begin{vmatrix} 0 & s-1 \\ E_0 \frac{s}{s^2 + \omega^2} e^{-s} & 2s \end{vmatrix} = -E_0 \frac{s(s-1)}{s^2 + \omega^2} e^{-s}$$

$$\Delta_2 = \begin{vmatrix} s & 0 \\ s+1 & E_0 \frac{s}{s^2 + \omega^2} e^{-s} \end{vmatrix} = E_0 \frac{s^2}{s^2 + \omega^2} e^{-s}$$

$$X(s) = -E_0 \frac{s^2 - s}{(s^2 + \omega^2)(s^2 + 1)} e^{-s} \quad \omega \neq 1, \omega \in \mathbb{R}$$

$$Y(s) = E_0 \frac{s^2}{(s^2 + \omega^2)(s^2 + 1)} e^{-s}$$

$$= \frac{1}{1-\omega^2} \left[-\frac{1}{s^2 + \omega^2} - \omega \frac{\omega}{s^2 + \omega^2} + \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1} \right]$$

$$\frac{s^2 - s}{(s^2 + \omega^2)(s^2 + 1)} = \frac{As + B}{s^2 + \omega^2} + \frac{C + D}{s^2 + 1} = \frac{1}{1-\omega^2} \frac{-s - \omega^2}{s^2 + \omega^2} + \frac{1}{1-\omega^2} \frac{s + 1}{s^2 + 1}$$

$$\lim_{s \rightarrow i} \frac{s^2 - s}{s^2 + \omega^2} = Ci + D$$

$$\frac{-1 - i}{-1 + \omega^2} = Ci + D \quad \frac{1 + i}{1 - \omega^2} = Ci + D$$

$$\lim_{s \rightarrow i\omega} \frac{-\omega^2 - i\omega}{1 - \omega^2} = Ai\omega + B$$

$$\frac{-\omega^2}{1 - \omega^2} = B$$

$$\frac{-\omega}{1 - \omega^2} = A\omega \Rightarrow$$

$$A = \frac{-1}{1 - \omega^2}$$

$$C = \frac{1}{1 - \omega^2}, \quad D = \frac{1}{1 - \omega^2}$$

$$\frac{s^2 - s}{(s^2 + \omega^2)(s^2 + 1)} = \frac{1}{1 - \omega^2} \left[\frac{-s - \omega^2}{s^2 + \omega^2} + \frac{s + 1}{s^2 + 1} \right] = \frac{1}{1 - \omega^2} \left[-\frac{s}{s^2 + \omega^2} - \frac{\omega^2}{s^2 + \omega^2} + \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1} \right]$$

$$x'(t) + y'(t) - y(t) = 0$$

$$x'(t) + x(t) + 2y'(t) = E_0 \omega j \sin(\omega t - \phi) h(t-1)$$

Ende 7.30

$$jX(s) + (s-1)Y(s) = 0$$

$$(j+1)X(s) + 2jY(s) = E_0 e^{-j} \frac{s}{s^2 + \omega^2}$$

$$\Delta = \begin{vmatrix} j & s-1 \\ j+1 & 2s \end{vmatrix} = j^2 + 1$$

$$\Delta_1 = \begin{vmatrix} 0 & s-1 \\ E_0 e^{-j} \frac{j}{j^2 + \omega^2} & 2s \end{vmatrix} = -E_0 \frac{j^2 - s}{j^2 + \omega^2} e^{-j}$$

$$\Delta_2 = \begin{vmatrix} j & 0 \\ 0 & E_0 \frac{s}{j^2 + \omega^2} e^{-j} \end{vmatrix} = E_0 \frac{j^2}{j^2 + \omega^2} e^{-j}$$

$$X(s) = -E_0 \frac{j^2 - s}{(j^2 + \omega^2)(j^2 + 1)} e^{-j}$$

$$x(t) = \frac{E_0}{1-\omega^2} [\omega j \sin(\omega t - \phi) + \omega \sin(\omega t - \phi) - \omega j \sin(\omega t - \phi)]$$

$$Y(s) = E_0 \frac{j^2}{(j^2 + \omega^2)(j^2 + 1)} e^{-j}$$

$$y(t) = \frac{E_0}{1-\omega^2} [\omega j \sin(\omega t - \phi) + \sin(\omega t - \phi)] h(t-1)$$

$$\frac{j^2 - s}{(j^2 + \omega^2)(j^2 + 1)} = \frac{Aj + B}{j^2 + \omega^2} + \frac{Cj + D}{j^2 + 1}$$

$$Ai\omega + Bi = \frac{-\omega^2 - i\omega}{1-\omega^2} \Rightarrow A = -\frac{1}{1-\omega^2}, B = -\frac{\omega}{1-\omega^2}$$

$$Ci + Di = \frac{-1 - i}{1-\omega^2} = \frac{1}{1-\omega^2} + i \frac{1}{1-\omega^2} \quad C = \frac{1}{1-\omega^2} \quad D = \frac{1}{1-\omega^2}$$

$$\frac{j^2 - s}{(j^2 + \omega^2)(j^2 + 1)} = -\frac{1}{1-\omega^2} \left(\frac{j}{j^2 + \omega^2} + \frac{\omega^2}{j^2 + \omega^2} \right) + \frac{1}{1-\omega^2} \left(\frac{j}{j^2 + 1} + \frac{1}{j^2 + 1} \right) \checkmark$$

$$X(s) = \frac{E_0}{1-\omega^2} \left[\frac{j}{j^2 + \omega^2} + \omega \frac{\omega}{j^2 + \omega^2} - \left(\frac{j}{j^2 + 1} + \frac{1}{j^2 + 1} \right) \right] e^{-j}$$

$$\frac{j^2}{(j^2 + \omega^2)(j^2 + 1)} = \frac{Aj + B}{j^2 + \omega^2} + \frac{Cj + D}{j^2 + 1} \quad Ai\omega + Bi = \frac{-\omega^2}{1-\omega^2} \quad A=0, B = \frac{-\omega^2}{1-\omega^2}$$

$$Ci + Di = \frac{-1}{1-\omega^2} = \frac{1}{1-\omega^2} \Rightarrow C=0, D=\frac{1}{1-\omega^2}$$

$$Y(s) = \frac{E_0}{1-\omega^2} \left[-\omega \frac{\omega}{j^2 + \omega^2} + \frac{1}{j^2 + 1} \right] e^{-j}$$

$$\frac{s^2}{(s^2 + \omega^2)(s^2 + 1)} = \frac{As + B}{s^2 + \omega^2} + \frac{Cs + D}{s^2 + 1} = -\frac{\omega}{s^2 + \omega^2} \frac{\omega}{s^2 + \omega^2} + \frac{1}{s^2 + \omega^2} \frac{1}{s^2 + 1}$$

$$A + \omega + B = \frac{-\omega^2}{s^2 + \omega^2}$$

$$A = 0, \quad B = -\frac{\omega^2}{s^2 + \omega^2} \quad \checkmark$$

$$\frac{-1}{s^2 + \omega^2} = Ci + Di \quad D = \frac{1}{s^2 + \omega^2} \quad C = 0$$

$$X(s) = \frac{E_0}{s - \omega} \left[-\frac{s}{s^2 + \omega^2} + \omega \frac{\omega}{s^2 + \omega^2} - \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1} \right] e^{-s}$$

$$Y(s) = \frac{E_0}{s + \omega} \left[-\omega \frac{\omega}{s^2 + \omega^2} + \frac{1}{s^2 + 1} \right] e^{-s}$$

$$x(t) = \frac{E_0}{s - \omega} \left[\cos(\omega t - 1) + \omega \sin(\omega t - 1) - \cos(t - 1) - \sin(t - 1) \right] h(t - 1)$$

$$y(t) = \frac{E_0}{s + \omega} \left[-\omega \sin(\omega t - 1) + \sin(t - 1) \right] h(t - 1)$$

$$I = \int_0^{\pi} \frac{\ln x}{x^4 + 1} dx$$

$$f(z) = \frac{\ln z}{z^4 + 1}$$

$$z^4 + 1 = 0$$

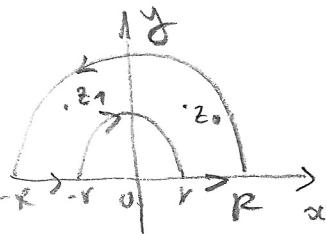
$$z = 0$$

$$z^4 = -1$$

$$z^4 = e^{i\pi}$$

$$z_k = e^{i \frac{i\pi + 2k\pi}{4}}$$

$$k=0, 1, 2, 3$$



1

$$z_0 = e^{i\pi/4} e^{i\pi t} \quad \left\{ \text{so with I present} \right.$$

$$z_1 = e^{i3\pi/4} e^{i\pi t}$$

$$z_2 = e^{i5\pi/4} \notin \text{int P}$$

$$z_3 = e^{i7\pi/4} \notin \text{int P}$$

$$z_4 = 0 \notin \text{int}$$

3

$$\oint f(z) dz = 2\pi i (\operatorname{Res}_{z=z_0} f(z) + \operatorname{Res}_{z=z_1} f(z))$$

2

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\ln z}{4z^3} \Big|_{z=e^{i\pi/4}} = \frac{i\pi/4}{4e^{i3\pi/4}} = i \frac{\pi}{16} e^{-i3\pi/4}$$

$$\operatorname{Res}_{z=z_1} f(z) = \frac{\ln z}{4z^3} \Big|_{z=e^{i7\pi/4}} = \frac{i7\pi/4}{4e^{i9\pi/4}} = i \frac{7\pi}{16} e^{-i\pi/4}$$

4

$$\operatorname{Re} f(z) + \operatorname{Re} \bar{f}(z) = i \frac{\pi}{16} \left(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right) + i \frac{7\pi}{16} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) = i \frac{\sqrt{2}\pi}{16} + \frac{\sqrt{2}\pi}{8}$$

$$\int_R^0 S + \int_0^R S + \int_{-R}^0 S + \int_{-R}^0 \bar{S} = 2\pi i \left(i \frac{\sqrt{2}\pi}{16} + \frac{\sqrt{2}\pi}{8} \right) \quad (1)$$

2

$$|z f(z)| = \left| \frac{z}{z^4 + 1} (\ln |z| + i \arg z) \right| \leq \frac{|z|}{|z|^4 - 1} \sqrt{\ln^2 |z| + \pi^2}$$

$$\text{ZEC}_R: |z f(z)| \leq \frac{R}{R^4 - 1} \sqrt{\ln^2 R + 4\pi^2} \rightarrow 0 \quad R \rightarrow \infty \quad (2)$$

5

$$\text{ZEC}_R: \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0 \quad \text{Jordan's lemma}$$

$$\text{ZEC}_r: |z f(z)| \leq \frac{r}{1-r^4} \sqrt{\ln^2 r + 4\pi^2} \rightarrow 0 \quad r \rightarrow 0 \quad (3)$$

$$\text{ZEC}_r: \lim_{r \rightarrow 0} \int_{C_r} f(z) dz = 0 \quad \text{Jordan's lemma}$$

Тако ю (1) и (3) доказано, а (2) доказано

$$\int_0^\infty \frac{\ln x}{x^4 + 1} dx + \int_0^\infty \frac{\ln x + i\pi}{x^4 + 1} e^{i\pi} dx = - \frac{\sqrt{2}\pi^2}{8} + i \frac{\sqrt{2}\pi^2}{4}$$

$$\text{а за бъде} \int_0^\infty \frac{\ln x}{x^4 + 1} dx = - \frac{\sqrt{2}\pi^2}{16} \quad \text{и} \quad \int_0^\infty \frac{dx}{x^4 + 1} = \frac{\pi}{2\sqrt{2}} \quad \checkmark$$

Ispit iz Matematike 3

Zadaci

1. (13) Odrediti ekstremne vrednosti funkcije $f(x, y) = x^2ye^{2x-y}$.
2. Dat je eliptički paraboloid $3x^2 + y^2 = 3 - z$.
- 1° (10) Izračunati površinski integral

$$I_1 = \iint_S (2xy - x^2) dy dz + (2yz - y^2) dz dx + (2zx - 2z^2) dx dy,$$

gde je S spoljna strana tela ograničenog datim paraboloidom i ravni $z = 0$.

- 2° (4+8) Direktnom metodom izračunati krivolonijski integral

$$I_2 = \oint_C y dx + z dy + x dz,$$

gde je kriva C presek datog paraboloida i paraboličkog cilindra $z = 2y^2$ pozitivno orijentisana ako se posmatra iz tačke $(0, 0, 3)$. Rezultat provjeriti primenom STOKESove formule.

Odseci OE, OF i OS - N. Cakić

3. (20) Kompleksnom integracijom izračunati $\int_0^{+\infty} \frac{\ln x}{x^4 + 1} dx$.
4. (15) Primenom \mathcal{L} -transformacije odrediti partikularno rešenje sistema diferencijalnih jednačina

$$\begin{aligned} x'(t) + y'(t) - y(t) &= 0 \\ x'(t) + x(t) + 2y'(t) &= f(t) \end{aligned}$$

koje zadovoljava početne uslove $x(0) = y(0) = 0$, gde je

$$f(t) = \begin{cases} 0, & t < 1 \\ E_0 \cos \omega(t-1), & t \geq 1 \end{cases}, \quad \omega \in \mathbb{R}, \omega \neq 1.$$

Odseci OT i OG - S. Ješić

3. (20) Kompleksnom integracijom izračunati $\int_0^{+\infty} \frac{x^2 - a^2}{x^2 + a^2} \frac{\sin x}{x} dx, \quad (a > 0)$.

- 4.(15) Funkciju $f(x) = \max(\sin x, 0)$ predstaviti FOURIERovim redom na intervalu $[-\pi, \pi]$, a zatim na osnovu dobijenog rezultata izračunati sumu brojnog reda $\sum_{n=1}^{+\infty} \frac{1}{4n^2 - 1}$.

МАТ. З / РЕШЕЊА

1. $f_x = 2xy(1+x)e^{2x-y}$ стационарн. тачке $N_1(0,y)$ цела y -оса

$$f_y = x^2(1-y)e^{2x-y} = 0 \quad M_2(-1,1)$$

$$f_{xx} = y(2+8x+4x^2)e^{2x-y} \quad \text{за } N_2(-1,1)$$

$$f_{xy} = 2x(1+x)(1-y)e^{2x-y} \quad A = f_{xx}(-1,1) = -2e^{-3} < 0$$

$$f_{yy} = x^2(y-2)e^{2x-y} \quad B = 0$$

$$C = -e^{-3}$$

$$AC - B^2 = 2e^{-6} > 0$$

$$\max f(-1,1) = e^{-3}$$

за $M_1(0,y)$ $A = f_{xx}(0,y) = 2ye^{-y}$

$$B = f_{xy}(0,y) = 0 \quad AC - B^2 = 0 \quad \text{, додатна испитиванја}$$

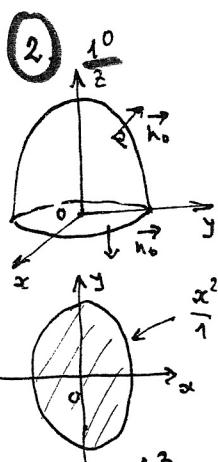
$$C = f_{yy}(0,y) = 0 \quad \text{ преко приредбада}$$

$$\Delta f(0,y) = f(\Delta x, y + \Delta y) - f(0, y) = \Delta x^2(y + \Delta y)e^{2\Delta x - (y + \Delta y)}$$

за $y > 0 \quad \Delta f(0,y) \geq 0$ за доволно мало Δy па и највиши нестроги минимум

за $y < 0 \quad \Delta f(0,y) \leq 0 \quad \text{нестроги максимум } f(0,y) = 0$

за $y = 0 \quad \Delta f(0,0) = \Delta x^2 \cdot \Delta y e^{2\Delta x - \Delta y} \begin{cases} \geq 0 & \Delta y \geq 0 \\ \leq 0 & \Delta y < 0 \end{cases} \quad \text{доказатво експоненцијално}$

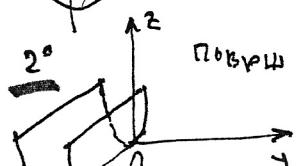


$$I_1 = -2 \iiint z \, dx \, dy \, dz \quad \text{применом формуле Остроградског}$$

$$\text{у виду по цилиндричне координате.} \quad \begin{cases} x = s \cos \varphi \\ y = \sqrt{3} s \sin \varphi \\ z = z \end{cases} \quad J = \sqrt{3} \pi$$

$$I_1 = -2 \iiint z \sqrt{3} s \cos \varphi \, ds \, d\varphi \, dz = -3\sqrt{3}\pi$$

$0 \leq \varphi \leq 2\pi$
 $0 \leq s \leq 1$
 $0 \leq z \leq 3(1-s^2)$



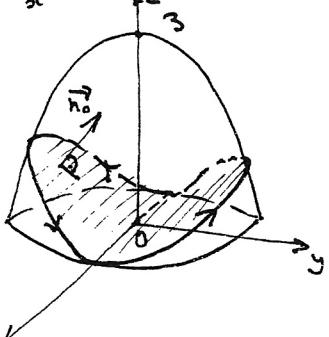
$$\text{Површи } z = 2y^2$$

ортогонална пројекција криве са равни Oxy је круж $x^2 + y^2 = 1$. С обзиром на узету оријентацију криве круж $x^2 + y^2 = 1$ је позитивно оријентисан. Параметарске једначине сују кривес

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = 2\sin^2 t = 1 - \cos 2t \end{cases} \quad 0 \leq t \leq 2\pi$$

$$I_2 = \int_0^{2\pi} [2\sin t (-\sin t) + (1 - \cos 2t) \cos t + \cos t (-2\sin t)] dt = -\pi$$

Где смо користили ортогоналност тригономет. ф-ја на $[0, 2\pi]$



Stokes - 6a epormyia

$$I_2 = - \iint_S (\cos \alpha + \cos \beta + \cos \gamma) dS$$

Следе S до површи парabolичног цилиндра $z = 2y^2$ који се ослања на криву C оријентисану унутрашњом нормалом $\vec{n}_0 = (\cos \alpha, \cos \beta, \cos \gamma)$ ако $\gamma < \pi/2$.

$$\leq: z = 2y^2 \Rightarrow z_x = 0, z_y = 4y$$

$$\cos \gamma = \frac{1}{\sqrt{1+16y^2}}, \quad \cos d > 0, \quad \cos \beta = \frac{-4y}{\sqrt{1+16y^2}}$$

Определите область проекции на плоскость Oxy для поверхности S , заданной уравнением $x^2 + y^2 \leq 1$.

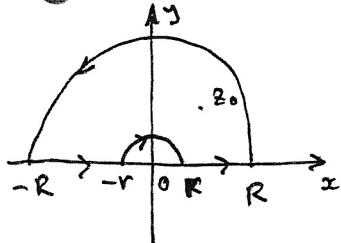
$$I_2 = - \iint_S \left(\frac{-4y}{\sqrt{1+16y^2}} + \frac{1}{\sqrt{1+16y^2}} \right) dS = - \iint_{x^2+y^2 \leq 1} \frac{-4y+1}{\sqrt{1+16y^2}} \sqrt{1+16y^2} dx dy$$

$$= + \underbrace{\iint_{x^2+y^2 \leq 1} y \, dx \, dy}_{= 0} - \underbrace{\iint_{x^2+y^2 \leq 1} \sin x \, dx \, dy}_{= \pi \cdot 1^2} = -\pi.$$

EE, OC, oφ

3) Определение интеграла при дробном

$$\oint_C f(z) dz, \quad f(z) = \frac{\ln z}{z^4 + 1}$$



$$\oint_C f(z) dz = 2\pi i \left(\operatorname{Res}_{z=z_0} f(z) + \operatorname{Res}_{z=z_1} f(z) \right) \quad (1)$$

$$z_0 = e^{i\pi/4} \in \text{int}\Gamma \quad \text{non I-peda}$$

$$z_1 = e^{i\pi/4} \in \text{int } \Gamma \quad \text{non T-polya}$$

Сингулярне таине $z=0 \notin \text{int} \Gamma$, $z_2 = e^{i\pi/4} \notin \text{int} \Gamma$, $z_3 = e^{i7\pi/4} \notin \text{int} \Gamma$

$$\operatorname{Re} f(z) = \frac{\ln z}{4z^3} \Big|_{z=e^{i\pi/4}} = i \frac{\pi}{16} \left(-\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right)$$

$$\left. \operatorname{Res}_{z=2} f(z) = \frac{\ln z}{4z^3} \Big|_{z=e^{i\pi/4}} = i \frac{3\pi}{16} \left(\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) \right\}$$

$$\oint_C f(z) = \int_r^R + \int_{C_R} + \int_{-R}^{-r} + \int_{C_r} \quad (2)$$

$$\text{Now to see } |z f(z)| \leq \frac{|z|}{|z|^4 - 1} \sqrt{\ln^2 |z| + 4\pi^2} = \frac{R}{R^4 - 1} \sqrt{\ln^2 R + 4\pi^2} \rightarrow 0 \quad R \rightarrow \infty$$

$$D \ni z \in \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0 \quad \text{Jordan's lemma}$$

$$|z f(z)| \leq \frac{r}{1-r^4} \sqrt{\ln^2 r + 4\pi^2} \rightarrow 0 \quad \text{as } r \rightarrow 0^+ \quad \Rightarrow \quad \lim_{r \rightarrow 0^+} \int_{C_r} f(z) dz = 0 \quad \text{Jordan's Lemma}$$

Ако неко професионално (1) и (2) и Јорданове креативнији пошто нукали да

$$\int_0^{+\infty} \frac{\ln x}{x^4 + 1} dx + \int_{-\infty}^0 \frac{\ln x e^{i\pi}}{x^4 + 1} e^{i\pi} dx = 2\pi i \left(\frac{j\sqrt{2}}{8} + i \frac{\pi\sqrt{2}}{16} \right)$$

Однако для вычисления

$$\int_0^{+\infty} \frac{dx}{x^4 + 1} dx = -\frac{\pi^2 \sqrt{2}}{16}$$

$$\text{наиболее} \int_0^{+\infty} \frac{dx}{x^4 + 1} = \frac{\pi \sqrt{2}}{4}$$

4)

$$f(t) = E_0 \cos \omega(t-1) h(t-1) \Rightarrow F(s) = L\{f(t)\} = E_0 e^{-s} \frac{s}{s^2 + \omega^2}$$

Применяя Z-变换а в системе Добицамо

$$sX(s) + (s-1)Y(s) = 0$$

$$(s+1)X(s) + 2sY(s) = E_0 \frac{s}{s^2 + \omega^2} e^{-s}$$

$$X(s) = -E_0 \frac{s^2 - s}{(s^2 + \omega^2)(s^2 + 1)} e^{-s}, \quad Y(s) = E_0 \frac{s^2}{(s^2 + \omega^2)(s^2 + 1)} e^{-s}$$

$$X(s) = \frac{E_0}{1 - \omega^2} \left[\frac{s}{s^2 + \omega^2} + \omega \frac{\omega}{s^2 + \omega^2} - \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1} \right] e^{-s}$$

$$Y(s) = \frac{E_0}{1 - \omega^2} \left[-\omega \frac{\omega}{s^2 + \omega^2} + \frac{1}{s^2 + 1} \right] e^{-s}$$

наze

$$x(t) = \frac{E_0}{1 - \omega^2} \left[\cos \omega(t-1) + \omega \sin \omega(t-1) - \cos(t-1) - \sin(t-1) \right] h(t-1)$$

$$y(t) = \frac{E_0}{1 - \omega^2} \left[-\omega \sin \omega(t-1) + \sin(t-1) \right] h(t-1)$$

МУ

Ispit iz Matematike 3

Zadaci

1. (11) Na sferi $x^2 + y^2 + z^2 = 1$ naći tačku M tako da je zbir kvadrata od nje do tačaka $A_1(1, -1, 0), A_2(-1, -1, 1), A_3(1, 1, 1)$ minimalan.
(Napomena: prirodu tačke M ispitati preko drugog diferencijala.)

2. 1° (13) Izračunati

$$\int_C (\sin xy + xy \cos xy) dx + x^2(\cos xy + 1) dy.$$

C je kriva AOB gde je \widehat{AO} polukrug $x^2 + y^2 = 2x$ ($y \geq 0$) od $A(2, 0)$ do $O(0, 0)$, a \widehat{OB} polukrug $x^2 + y^2 = x$ ($y \geq 0$) od $O(0, 0)$ do $B(1, 0)$.

2° (11) Izračunati

$$\iiint_D z(x^2 + y^2) dx dy dz$$

gde je D ograničena površima $z = x^2 + y^2, z = 2 - x^2 - y^2$.

3.(20) Kompleksnom integracijom izračunati vrednost realnog integrala

$$\int_0^{+\infty} \frac{\sin \alpha x}{x^5 + \alpha x} dx,$$

gde je $\alpha > 0$ realan parametar.

4.(15) Funkciju $f(t) = \max(\sin t, 0)$ razviti u Furijeov red, a zatim izračunati sumu $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.

Teorijska pitanja

Odseci OT i OG - S. Ješić

1.(15) MOORE-ova teorema o razmeni dva limesa. Iskaz i dokaz.

2.(15) Jednoznačne i više značne funkcije kompleksne promenljive. Navesti primere i odrediti sve vrednosti funkcije $f(z) = \ln z$ u tački $z = -i$.

Odseci OE, OF i OS - N. Cakić

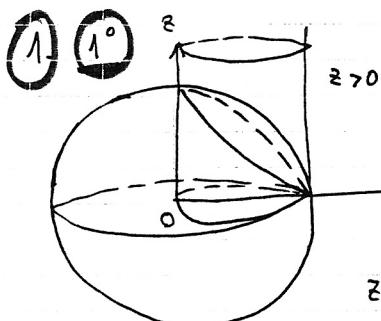
1.(15) Implicitna funkcija.

2.(15) Izračunavanje određenih integrala primenom računa ostataka.

Ispit traje 180 minuta. Na naslovnoj strani vežbanke obavezno precrtati brojeve zadataka koji nisu rađeni. Ispit je položen ukoliko kandidat sakupi barem 35 poena na zadacima i barem 15 poena na teorijskim pitanjima.

Beograd, 1. septembar, 2012.

МАТ. 3 (решења)



Трапезна површина је

$$P = \iint_D \sqrt{1+z_x^2+z_y^2} \, dx \, dy$$

$$\alpha \quad \text{где } z = \sqrt{a^2 - x^2 - y^2} \quad \text{и } D: x^2 + y^2 \leq ax$$

$$z_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, \quad z_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}} \Rightarrow \sqrt{1+z_x^2+z_y^2} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

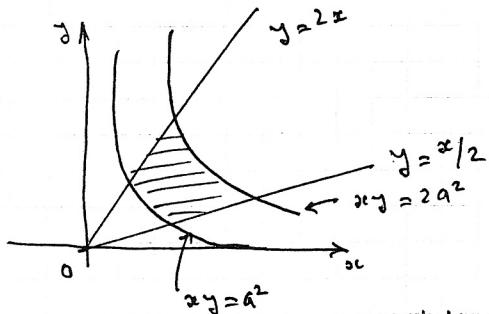
$$P = \iint_{D'} \frac{a}{\sqrt{x^2+y^2} \leq ax} \, dx \, dy, \quad \text{полярне координате}$$

$$x = s \cos \varphi, \quad y = s \sin \varphi$$

$$P = \iint_{0 \leq s \leq a \cos \varphi} \frac{a}{\sqrt{a^2 - s^2}} s \, ds \, d\varphi = a \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{a \cos \varphi} \frac{s}{\sqrt{a^2 - s^2}} \, ds = a^2 (\pi - 2)$$

2° Трапезна здравима је $V = \iint_D (x^2 + y^2) \, dx \, dy$, где је D једногранник обласја и да слични

$$\text{смеште: } u = xy, \quad v = \frac{y}{x}, \quad \text{тада } a^2 \leq u \leq 2a^2$$



$$\frac{1}{2} \leq V \leq 2$$

јакобијан

$$J = \frac{D(x,y)}{D(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{2v} > 0$$

$$u^v = y^2, \quad x = \frac{u}{v} \Rightarrow x^2 = \frac{u^2}{v^2} = \frac{uv}{v^2} = \frac{u}{v} = \frac{u}{\frac{y}{x}} = \frac{u^2}{y^2}$$

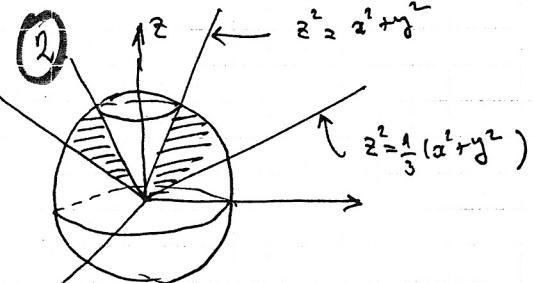
$$\text{тада } V = \iint_{a^2 \leq u \leq 2a^2} (uv + \frac{u}{v}) \cdot \frac{1}{2v} \, du \, dv = \frac{1}{2} \int_{a^2}^{2a^2} u \, du \int_{1/2}^2 \left(1 + \frac{1}{v^2}\right) \, dv = \frac{9}{4} a^4$$

$$\frac{1}{2} \leq V \leq 2$$

$$I = \iiint_D z^3 \, dx \, dy \, dz$$

области по којој се врши интеграција је једногранник

$$\begin{cases} x = s \cos \varphi \cos \theta \\ y = s \sin \varphi \cos \theta \\ z = s \sin \theta \end{cases} \quad J = s^2 \cos \theta$$



тада се обласј D пресликава у облику θ отприликом:

$$x^2 + y^2 = z^2 \Rightarrow \theta = \pi/4 \quad (z > 0)$$

$$x^2 + y^2 = 3z^2 \Rightarrow \theta = \pi/6 \quad (z > 0)$$

$$I = \iiint_D s^3 \sin^3 \theta \cdot s^2 \cos \theta \, ds \, d\varphi \, d\theta = \int_0^{2\pi} d\varphi \int_{\pi/6}^{\pi/4} \sin^3 \theta \cos \theta \, d\theta \int_0^R s^5 \, ds$$

$$= \frac{\pi}{64} R^6.$$

Ispit iz Matematike 3

Zadaci

1. (12) Odrediti ekstremne vrednosti funkcije $f(x, y) = 7x^2 + 8xy + y^2$ pod uslovom $x^2 + y^2 = 1$.

2. Date su površi $S_1 : \frac{x^2}{2} + \frac{y^2}{3} = 2z$, $S_2 : \left(\frac{x^2}{4} + \frac{y^2}{9}\right)^2 = \frac{x^2}{4} - \frac{y^2}{9}$ i $S_3 : z = \frac{y}{3}$.

1° (13) Izračunati površinu dela površi S_1 koja se nalazi unutar površi S_2 .

2° (12) Pomoću Stokes-ove formule izračunati integral

$$I = \oint_c y \, dx + z \, dy + x \, dz,$$

kriva c je presek površi S_1 i S_3 pozitivno orijentisana ako se posmatra iz tačke $(0, 1, 0)$.

Odseci OE, OF i OS - N. Cakić

3. (16) Primenom kompleksne integracije izračunati integral

$$I = \int_0^{2\pi} \frac{\cos 3x}{3 - 2 \cos x} \, dx.$$

4. (19) Koristeći Laplace-ovu transformaciju rešiti diferencijalnu jednačinu

$$x''(t) + 4x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 1,$$

$$\text{gde je } f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}.$$

Odseci OT i OG - S. Ješić

3. (20) Kompleksnom integracijom izračunati vrednost integrala

$$\int_0^{+\infty} \frac{\sin ax}{x(1+x^4)} \, dx, \quad (a > 0).$$

4.(15) Funkciju $f(x) = \frac{1}{2} - \frac{\pi}{4} \sin x$ predstaviti kosinusnim FOURIERovim redom na intervalu $[0, \pi]$. Na osnovu dobijenog rezultata odrediti sumu reda

$$\sum_{k=1}^{+\infty} \frac{(-1)^k}{4k^2 - 1}.$$

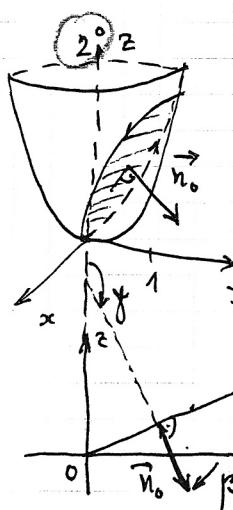
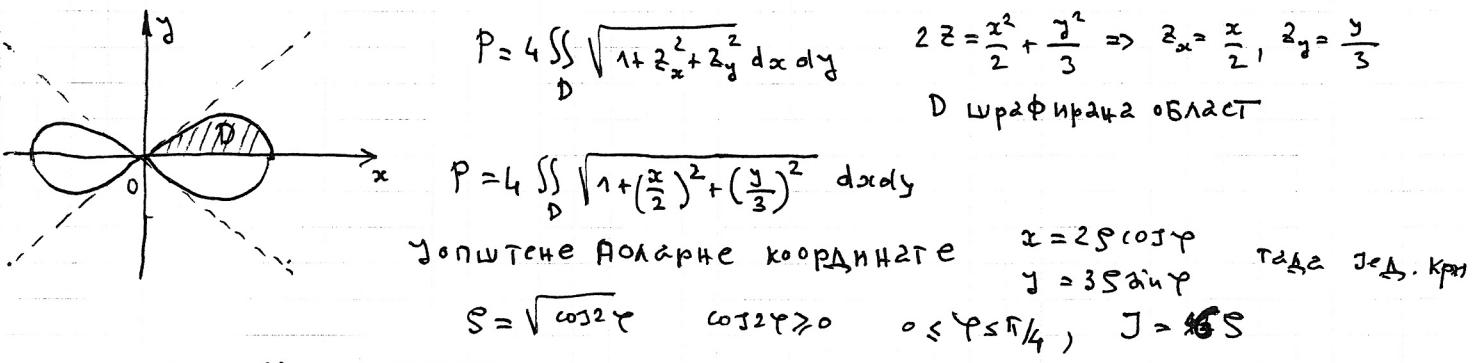
MAT. 3.

1. Результат: $M_{1,2} \left(\pm \frac{2}{\sqrt{5}}, \pm \frac{1}{\sqrt{5}} \right)$ строги лок. макс $f(M_{1,2}) = 9$

$M_{3,4} \left(\pm \frac{1}{\sqrt{5}}, \mp \frac{2}{\sqrt{5}} \right) \rightarrow \min f(M_{3,4}) = -1.$

2. 1° $\frac{x^2}{4} - \frac{y^2}{9} \geq 0 \Leftrightarrow |y| \leq \frac{3}{4}|x|$ кривая $\left(\frac{x^2}{4} + \frac{y^2}{9} \right)^2 = \frac{x^2}{4} - \frac{y^2}{9}$ симметрична

координатам по x -оси, y -оси, коорд. по x -оси



$$I = \iint_S \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} dS = \iint_S (\cos \alpha + \cos \beta + \cos \gamma) dS$$

Где S - это поверхность S_3 ориентированной нормалем \vec{n} по правилу

Танк $(0, 1, 0)$ Тогда $\pi/2 < \gamma < \pi, -\pi/2 < \beta < 0 \Rightarrow \cos \gamma < 0$

$$\cos \alpha = \frac{1}{-\sqrt{1+z_x^2+z_y^2}} = -\frac{3}{\sqrt{10}}, \cos \beta = 0, \cos \gamma = \frac{1}{\sqrt{10}}$$

$$z = \frac{y}{3}, z_x = 0, z_y = -\frac{1}{3}$$

$$I = \frac{2}{\sqrt{10}} \iint_S dS = \frac{2}{\sqrt{10}} \iint_D \sqrt{1+z_x^2+z_y^2} dx dy$$

$\Gamma_A \in \mathbb{C} \setminus D$ огранич. кривом $\frac{x^2}{2} + \frac{y^2}{3} = 2y$, т.е. эллипсом $\frac{x^2}{2/3} + \frac{(y-1)^2}{1} = 1$

$$I = \frac{2}{\sqrt{10}} \iint_D \sqrt{1+\left(\frac{1}{3}\right)^2} dx dy = \frac{2}{\sqrt{10}} \cdot \frac{\sqrt{10}}{3} \iint_D dx dy = \frac{2}{3} P(D) = \frac{2}{3} \underbrace{\frac{2}{3} \cdot 1 \cdot \pi}_{\text{поверх. эллипс}}$$

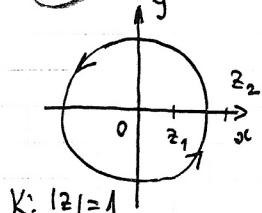
$$I = \left(\frac{2}{3}\right)^{3/2} \pi$$

$$\Gamma + J = \int_0^{2\pi} \frac{\cos 3x + i \sin 3x}{3 - 2 \cos x} dx = \int_0^{2\pi} \frac{e^{i 3x}}{3 - 2 \cos x} dx = \frac{1}{-i} \oint_{|z|=1} \frac{z^3}{z^2 - 3z + 1} dz$$

$$= -\frac{1}{i} \operatorname{Res}_{z=z_1} f(z), \quad \Gamma_A \in \mathbb{C} \quad f(z) = \frac{z^3}{z^2 - 3z + 1} dz, \quad z_1 = \frac{3 - \sqrt{5}}{2} \in \operatorname{int} K$$

$$= -\frac{1}{i} 2\pi i \frac{z^3}{(z^2 - 3z + 1)'} \Big|_{z=z_1} = -\frac{1}{i} 2\pi i \left(-\frac{1}{\sqrt{5}} \left(\frac{3 - \sqrt{5}}{2} \right)^3 \right) = \frac{2\pi}{\sqrt{5}} \left(\frac{3 - \sqrt{5}}{2} \right)^3$$

$$\Gamma_A \in \int_0^{2\pi} \frac{\cos 3x}{3 - 2 \cos x} dx = \frac{2\pi}{\sqrt{5}} \left(\frac{3 - \sqrt{5}}{2} \right)^3$$



$$4 \quad \mathcal{L}\{x''(t)\} + 4 \mathcal{L}\{x(t)\} = \mathcal{L}\{f(t)\}, \quad f(t) = h(t) - h(t-1)$$

$t \rightarrow h(t)$ - Heavy side-side функц.

$$\mathcal{L}\{x''(t)\} = s^2 X(s) - s x(0) - x'(0) = s^2 X(s) - 1$$

$$\mathcal{L}\{x(t)\} = X(s)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{h(t) - h(t-1)\} = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$n_2 \quad \text{re} \quad s^2 X(s) - 1 + X(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$X(s) = \frac{1}{s^2+4} + \frac{1}{s} \frac{1}{s^2+4} - e^{-s} \frac{1}{s} \cdot \frac{1}{s^2+4}$$

$$\frac{1}{s(s^2+4)} = \frac{1}{s^2(s^2+4)} = \frac{1}{4} \frac{s(s^2+4-s^2)}{s^2(s^2+4)} = \frac{1}{4} \left(\frac{1}{s} - \frac{1}{s^2+4} \right)$$

$$X(s) = \frac{1}{2} \cdot \frac{2}{s^2+2^2} + \frac{1}{4} \left(\frac{1}{s} - \frac{1}{s^2+2^2} \right) - \frac{1}{4} \left(\frac{1}{s} - \frac{1}{s^2+2^2} \right) e^{-s}$$

↓

$$x(t) = \frac{1}{2} \sin 2t + \frac{1}{4} \left(h(t) - \overset{=1}{\cos 2t} \right) h(t) - \frac{1}{4} (1 - \cos 2(t-1)) h(t-1)$$

МЖ