1. Sistemi obicnih d.j. Teorema o egzistenciji

t$\in (\alpha,\beta) \subset \boldsymbol{R}$ (eventualno **C**)

$x_i(t), i = 1, \dots, n$ $\dot{x}_i(t) = \frac{dx_i}{dt}$ $\ddot{x}_i(t) = \frac{d^2x_i}{dt^2}$

$f_1(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = 0$ $f_2(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = 0$

Normalni sistem d.j. (od n obicnih d.j.)

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2, \dots, x_n)$$
$$\frac{dx_2}{dt} = f_2(t, x_1, x_2, \dots, x_n)$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_n(t, x_1, x_2, \dots, x_n)$$ => **SISTEM ***

$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n); i = 1, \dots, n$ t$\in (\alpha,\beta)$

Def. Pod resenjem (integralom) sistema * podrazumeva se skup f-ja $x_1 = \varphi_1(t), x_2 = \varphi_2(t), \dots, x_n = \varphi_n(t); t\epsilon(\alpha,\beta)$, tj. $x_i = \varphi_i(t)$, i=1,…,n, ukoliko je $\frac{d\varphi_1}{dt} = f_1(t, \varphi_1, \varphi_2, \dots, \varphi_n)$, i=1,…,n.

Def. Resenje siistema * oblika $x_i = \varphi_i(t, C_1, \dots, C_n), i = 1, \dots, n$ gde su C_i proizvoljne nezavisne konstante naziva se *opste resenje sistema **.

Ako se u opstem resenju na bilo koji nacin odredi bar jedna konstanta od C_i, tada se takvo resenje naziva partikularno resenje sistema (partikularni integral sistema).

Def. (Couchy) Pod Kosijevim pocetnim uslovima podrazumevamo skup jednakosti $x_1(t_0) = x_{10,}$

$x_2(t_0) = x_{20,}$…,$x_n(t_0) = x_{n0,}[x_i(t_0) = x_{i0,}\ i = 1, \dots, n]$ gde su t_0 i x_{i0} zadati realni brojevi. Resenje sistema * koje zadovoljava Kosijeve pocetne uslove naziva se kosijevo resenje sistema *.

Sistem * mozemo napisati u obliku:

$\frac{dx_1}{f_1(t,x_1,\dots,x_n)} = \frac{dx_2}{f_2(t,x_1,\dots,x_n)} = \dots = \frac{dx_n}{f_n(t,x_1,\dots,x_n)} (= dt)$ --> simetrican oblik sistema

Matricni oblik

Sistem * ⇔ $\frac{dX}{dt} = F(t, X), t \in (\alpha,\beta)$

X=$\begin{bmatrix} x_1(t) \\ . \\ . \\ x_n(t) \end{bmatrix}$ → $\frac{dX}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ . \\ . \\ \frac{dx_n}{dt} \end{bmatrix}$, F=$\begin{bmatrix} f_1(t, x_1, x_2, \dots, x_n) \\ . \\ . \\ f_n(t, x_1, x_2, \dots, x_n) \end{bmatrix}$

<u>Teorema o egzistenciji resenja:</u> Neka je sistem *dat u formalnom obliku. Trazi se partikularno resenje $x_i = \varphi_i(t)$ $(i = 1, \dots, n)$ koje zadovoljava Kosijeve uslove. Ako su f-je $f_i(t, x_1, x_2, \dots, x_n)$ (kao f-je n+1 promenljive) neprekidne u nekoj okolini tacke $M_0(t_0, x_{10}, \dots, x_{n0})$ i ako su parcijalni izvodi $\frac{\partial f_i}{\partial x_k}$ $(i = 1, \dots, n,\ k = 1, \dots, n)$ neprekidni u toj okolini, tada normalni sistem jednacina * ima jedinstveno resenje $x_i = \varphi_i(t)$, $(i = 1, \dots, n)$ koje zadovoljava Kosijeve uslove.

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2. <u>Prvi integrali normalnog sistema d.j.</u>

Pretpostavimo da je ($x_1 = \varphi_1(t)$, $x_2 = \varphi_2(t)$, …, $x_n = \varphi_n(t)$) → jedno resenje sistema *
Neka je ovo bilo koje resenje datog sistema *. Neka je $\Psi(t, x_1, x_2, \dots, x_n) = C$ data relacija gde je C proizvoljna konstanta koja ne zavisi od x_i i od t. Ako je $\Psi(t, \varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)) = C_0$, gde je C_0 fiksirana vrednost parametra C, tada se ova relacija naziva <u>prvi integral</u> normalnog sistema *.
U opstem slucaju za sistem * mozemo dobiti n prvih integrala $\Psi_1(t, x_1, x_2, \dots, x_n)$=$C_1$ …….
$\Psi_1(t, x_1, x_2, \dots, x_n) = C_n$ i potreban i dovoljan uslov da je pomocu ovih n prvih integral definisano opste resenje normalnog sistema * je

$$\begin{vmatrix} \frac{\partial \Psi_1}{\partial x_1} & \frac{\partial \Psi_1}{\partial x_2} & \dots & \frac{\partial \Psi_1}{\partial x_n} \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial \Psi_n}{\partial x_1} & \frac{\partial \Psi_n}{\partial x_2} & \dots & \frac{\partial \Psi_n}{\partial x_n} \end{vmatrix} \neq 0 \rightarrow \frac{D(\Psi_1, \Psi_{2,\dots}\Psi_n)}{D(x_{1,}x_2, \dots, x_n)}$$

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3. <u>Metoda diferenciranja i eliminacije i metoda integrirajucih kombinacija (metoda prvih integrala)</u>

x,y,z;

I $\quad \frac{dx}{dt} = f_1(t, x, y, z)$

II $\quad \frac{dy}{dt} = f_2(t, x, y, z)$

III $\quad \frac{dz}{dt} = f_3(t, x, y, z)$

I → $\frac{d^2x}{dt^2} = \frac{\partial f_1}{dt} + \frac{\partial f_1}{dx} \cdot \frac{\partial x}{\partial t} + \frac{\partial f_1}{dy} \cdot \frac{\partial y}{\partial t} + \frac{\partial f_1}{\partial z} \cdot \frac{\partial z}{\partial t}$

$\frac{d^2x}{dt^2} = \overline{f_1}(t, x, y, z)$
$\frac{d^3x}{dt^3} = \overline{\overline{f_1}}(t, x, y, z)$ i $\frac{dx}{dt} = f_1(t, x, y, z)$

Od zaokruzenih izdvojicemo y i z iz bilo koje 2 jednacine i zamenicemo u trecu

- ➔ F(t,x,ẋ,ẍ)=0 → x=$\Phi(t, C_1, C_2, C_3)$

Metod prvih integrala

$$\frac{dx_1}{f_1(t,x_1,\dots,x_n)} = \frac{dx_2}{f_2(t,x_1,\dots,x_n)} = \cdots = \frac{dx_n}{f_n(t,x_1,\dots,x_n)}$$

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \frac{\lambda a+\mu b+\nu c}{\lambda A+\mu B+\nu C} \qquad \frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{dx+dy+dz}{0}$$

$$xdx + ydy + zdz = \frac{1}{2}d(x^2 + y^2 + z^2)$$

$$yzdx + xzdy + xydz = d(xyz)$$

- ⇨ d(x+y+z) = 0 => x+y+z=C_1

$$=\frac{xdx+ydy+zdz}{xy-xz+yz-xy+xz-yz} = \frac{\frac{1}{2}d(x^2+y^2+z^2)}{0} => x^2 + y^2 + z^2 = C_2$$

II logika: z=C_1-x-y

$$\frac{dx}{y-C_1+x+y} = \frac{dy}{C_1-x-y-x} \qquad (C_1 - 2x - y)dx + (C_1 - x - 2y)dy = 0$$

$$(C_1 - 2x - y) = P(x, y) \quad (C_1 - x - 2y) = Q(x, y)$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} => \ -1 = -1 \ \surd$$

- ➔ Postoji f-ja u(x,y) tako da je du=Pdx+Qdy

$$u(x, y) = \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy \qquad u(x, y) = C_2$$

$$u(x, y) = \int_{x_0}^{x}(C_1 - 2x - y)dx + \int_{y_0}^{y}(C_1 - x_0 - 2y)dy = \ C_1x - x^2 - xy\Big|_{x_0}^{x} + C_1y - x_0y - y^2\Big|_{y_0}^{y} = C_1x - x^2 - xy - C_1x_0 + x_0^2 + x_0y + C_1y - x_0y - y^2 - C_1y_0 + x_0y_0 + y_0^2 = C_1x - x^2 - xy + C_1y - y^2 - C_1x_0 + x_0^2 - C_1y_0 + x_0y_0 + y_0^2 = (x + y + z)x - x^2 - xy + (x + y + z)y - y^2 - \bar{C} = xz + yz + \bar{C}$$

$$x^2 + y^2 + z^2 = C_2$$

III nacin:

z=C_1-x-y (ovo sledi iz prvog integrala)

$$\frac{dx}{x(y-C_1+x+y)}=\frac{dy}{y(C_1-x-y-x)}$$

$y(C_1-2x-y)dx+x(C_1-x-2y)dy=0$

$\frac{\partial P}{\partial y}=C_1-2x-2y;\ \frac{\partial Q}{\partial x}=C_1-2x-2y$ leva strana jednaka desnoj

$=> \boxed{u=C_2}$

$u(x,y)=\int_{x_0}^{x}P(x,y)dx+\int_{y_0}^{y}Q(x_0,y)dy=\int_{x_0}^{x}y(C_1-2x-y)dx+\int_{y_0}^{y}x_0(C_1-x_0-2y)dy=C_1xy-x^2y-y^2x\Big|_{x_0}^{x}+C_1x_0y-x_0^2y-x_0y^2\Big|_{y_0}^{y}=C_1xy-x^2y-xy^2-C_1x_0y+x_0^2y+x_0y^2+C_1x_0y-x_0y^2-x_0^2y-C_1x_0y_0+x_0^2y_0+x_0y_0^2$

$u=C_1xy-x^2y-xy^2+const.$

$(x+y+z)xy-x^2y-xy^2=C_2$

$\boxed{xyz=C_2}$

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4. Linearni sistemi oblika d.j.

$\frac{dx}{dt}=\sin t\ \cdot x(t)+(t^2+1)y(t)+e^t-1$

$\frac{dy}{dt}=2x(t)+(t-2)y(t)+\frac{1}{t^2+1}$

t$\in(\alpha,\beta)\ \ t\to\ x_i(t),\quad i=1,\dots,n$

$\frac{dx_1}{dt}=a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n+f_1(t)$

$\frac{dx_2}{dt}=a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n+f_2(t)$

.

.

$\frac{dx_n}{dt}=a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n+f_n(t)$ *linearni sistem dif. jedn.*

$\frac{dx_1}{dt}=\sum_{i=1}^{n}a_{1i}x_i(t)+f_1(t)$ i tako vazi i za dx_n i za ostale jednacine.

$$\frac{dx_j}{dt} = \sum_{i=1}^{n} a_{ji} x_i(t) + f_j(t), \quad j = 1, \dots, n$$

$f_i(t)$ su slobodni clanovi, a_{ji} su koeficijenti sistema. Ako je $f_j(t) = 0$ sistem je homogen, a inace je nehomogen.

Matricni zapis:

$$X=\begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_n(t) \end{bmatrix}=[x_1(t) \quad x_2(t) \quad \dots \quad x_n(t)]^T$$

$$\frac{dX}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ . \\ . \\ \frac{dx_n}{dt} \end{bmatrix}=\begin{bmatrix} \frac{dx_1}{dt} & \dots & \frac{dx_n}{dt} \end{bmatrix}^T$$

$$A(t)=\begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ . & . & . & . \\ . & . & . & . \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}=[a_{ij}]_{n \times n} \qquad F=\begin{bmatrix} f_1(t) \\ . \\ . \\ f_n(t) \end{bmatrix}=[f_1(t) \quad \dots \quad f_n(t)]^T$$

$\frac{dX}{dt} = A \cdot X + F \; za \; nehom. odn. \frac{dX}{dt} = A \cdot X \; za \; hom. jedn.$

Na osnovu Kosijeve teoreme linearni sistem ima resenje ako su koeficijenti sistema $a_{ij}(t)$ kao i sl. clanovi $f_i(t)$ neprekidne funkcije. Pod ovim uslovima psotoji jedno i samo jedno resenje sistema $x_1=x_1(t)$, $x_2=x_2(t)$, ..., $x_n=x_n(t)$ koje zadovoljava pocetne uslove:

$x_1(t_0)=x_{10}$, $x_2(t_0)=x_{20}$, ..., $x_n(t_0)=x_{n0}$ pri cemu su $t_0 \in (\alpha, \beta)$ i x_{i0} takvi brojevi.

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5. Neka je dat homogeni sistem $\frac{dX}{dt} = A \cdot X$ i neka su resenja ovog sistema matrice $X_1=\begin{bmatrix} x_{11}(t) \\ . \\ . \\ x_{n1}(t) \end{bmatrix}$, $X_2=\begin{bmatrix} x_{12}(t) \\ . \\ . \\ x_{n2}(t) \end{bmatrix}$, ..., $X_n=\begin{bmatrix} x_{1n}(t) \\ . \\ . \\ x_{nn}(t) \end{bmatrix}$; $X_k=\begin{bmatrix} x_{1k}(t) \\ . \\ . \\ x_{nk}(t) \end{bmatrix}$, k=1, ..., n i ima **n** resenja. Ako su matrice X_k resenja datog homogenog sistema, tada je resenje sistema i matrica $X=\sum_{k=1}^{n} C_k X_k \; (C_k = const.)$

Dokaz: $\frac{dX_k}{dt} = A \cdot X_k$ $\quad \frac{dX}{dt} = \frac{d}{dt}\sum_{k=1}^{n} C_k X_k = \sum_{k=1}^{n} C_k \frac{dX_k}{dt} = \sum_{k=1}^{n} C_k \cdot A \cdot x_k = A \cdot \sum_{k=1}^{n} C_k x_k = A \cdot X$

Bice opste resenje ako su x_k linearno nezavisni. Neka je $\sum_{k=1}^{n} \lambda_k x_k$ linearna kombinacija matrica. Kazacemo da su matrice X_k linearno nezavisne, tj. da se ni jedna od njih ne moze izraziti kao linearna kombinacija ostalih ukoliko vazi implikacija $\sum_{k=1}^{n} \lambda_k x_k = 0 => \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$

U protivnom kazemo da su matrice linearno zavisne.

W= $\begin{bmatrix} x_{11}(t) & x_{12}(t) & \dots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ . & . & . & . \\ . & . & . & . \\ x_{n1}(t) & x_{n2}(t) & \dots & x_{nn}(t) \end{bmatrix}$ Ako su koeficijenti sistema a_{ij} neprekidne funkcije i neka su matrice X_k (k=1, ..., n) resenja datog sistema (ili njeni partikularni integrali). Da bi part. integrali X_k bili linearno nezavisni potrebno je i dovoljno da determinanta Vronskog bude razl. od nule. Pod ovim uslovima opste resenje homogenog sistema je oblika $X(t) = \sum_{k=1}^{n} C_k x_k(t)$.

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6. Fundamentalni sistem resenja i homogeno resenje sistema d.j.

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

.

.

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \qquad a_{ij} = a_{ij}(t)$$

Neka su X_k partikularni inegrali ovog sistema (homogenog sistema) X_k. k=1, ..., n.

$X_1 = \begin{bmatrix} x_{11}(t) \\ . \\ . \\ x_{n1}(t) \end{bmatrix}$, $X_2 = \begin{bmatrix} x_{12}(t) \\ . \\ . \\ x_{n2}(t) \end{bmatrix}$, ..., $X_n = \begin{bmatrix} x_{1n}(t) \\ . \\ . \\ x_{nn}(t) \end{bmatrix}$

Proizvoljan niz od n linearno nezavisnih partikularnih integrala X_k naziva se fundamentalni sistem resenja homogenog sistema. Njihova linearna kombinacija X(t)= $\sum_{k=1}^{n} C_k x_k(t)$ je opste resenja homogenog sistema.

X(t)=$\begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_n(t) \end{bmatrix}$, X(t)= $\sum_{k=1}^{n} C_k x_k(t)$ <=> $\begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_n(t) \end{bmatrix} = C_1 \cdot \begin{bmatrix} x_{11}(t) \\ . \\ . \\ x_{n1}(t) \end{bmatrix} + C_2 \cdot \begin{bmatrix} x_{12}(t) \\ . \\ . \\ x_{n2}(t) \end{bmatrix} + \cdots + C_n \cdot$
$\begin{bmatrix} x_{1n}(t) \\ . \\ . \\ x_{nn}(t) \end{bmatrix}$

$x_1(t) = C_1 x_{11}(t) + C_2 x_{12}(t) + \cdots + C_n x_{1n}(t)$

$x_2(t) = C_1 x_{21}(t) + C_2 x_{22}(t) + \cdots + C_n x_{2n}(t)$

...

$x_n(t) = C_1 x_{n1}(t) + C_2 x_{n2}(t) + \cdots + C_n x_{nn}(t)$

za $n = 3$

X=$\sum_{k=1}^{3} C_k X_k$

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = C_1 \cdot \begin{bmatrix} x_1(t) \\ y_1(t) \\ z_1(t) \end{bmatrix} + C_2 \cdot \begin{bmatrix} x_2(t) \\ y_2(t) \\ z_2(t) \end{bmatrix} + C_3 \cdot \begin{bmatrix} x_3(t) \\ y_3(t) \\ z_3(t) \end{bmatrix}$$

$x(t) = C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t)$

$y(t) = C_1 y_1(t) + C_2 y_2(t) + C_3 y_3(t)$

$z(t) = C_1 z_1(t) + C_2 z_2(t) + C_3 z_3(t)$

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7. Opste resenje nehomogenog sistema. Lagranzov metod devijacije konstanti

$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1(t)$

$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2(t)$

.

.

$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n(t)$ $\qquad \frac{dX}{dt} = A \cdot X + F$

Opste resenje nehomogenog sistema je oblika $x(t) = X_h(t) + X^*(t)$ gde je $X_h(t)$ opste resenje homogenog sistema, a $X^*(t)$ je jedno partikularno resenje nehomogenog sistema.

Neka je $X_h(t)$ opste res. homogenog sistema oblika $\sum_{k=1}^{n} C_k x_k(t)$. Pretpostavimo da su $C_k = C_k(t)$ pa imamo $X_h(t) = \sum_{k=1}^{n} C_k x_k(t)$, pa je $\frac{dX_k}{dt} = A \cdot X_k + F$ <=> $\sum_{k=1}^{n} C_k' \cdot X_k + C_k X_k' = A \cdot \sum_{k=1}^{n} C_k x_k + F$

⇨ $\sum_{k=1}^{n} C_k'(t) X_k(t) = F(t)$

za n=3

$\sum_{k=1}^{3} C_k'(t)X_k(t) = F(t)$

$C_1'(t)\cdot\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} + C_2'(t)\cdot\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix} + C_3'(t)\cdot\begin{bmatrix}x_3\\y_3\\z_3\end{bmatrix} = \begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix}$

$x_1C_1'(t) + x_2C_2'(t) + x_3C_3'(t) = f_1$

$y_1C_1'(t) + y_2C_2'(t) + y_3C_3'(t) = f_2$

$z_1C_1'(t) + z_2C_2'(t) + z_3C_3'(t) = f_3$

D=W(t)= $\begin{vmatrix}x_1 & x_2 & x_3\\y_1 & y_2 & y_3\\z_1 & z_2 & z_3\end{vmatrix} \neq 0$ $\quad D_{C_1'} = \begin{vmatrix}f_1 & x_2 & x_3\\f_2 & y_2 & y_3\\f_3 & z_2 & z_3\end{vmatrix}, D_{C_2'} = \begin{vmatrix}x_1 & f_1 & x_3\\y_1 & f_2 & y_3\\z_1 & f_3 & z_3\end{vmatrix},$

$D_{C_3'} = \begin{vmatrix}x_1 & x_2 & f_1\\y_1 & y_2 & f_2\\z_1 & z_2 & f_3\end{vmatrix}$

Na osnovu Kramerove teoreme postoji resenje $C_k'(t) = \frac{D_{C_k'(t)}}{W(t)} = \varphi_k(t) \quad k = 1, \dots, n$

⇨ $C_k(t) = \int \varphi_k(t)dt + C_k = \phi_k(t) + C_k$ --> zamenimo u $X_h(t)$ i dobijamo:

opste resenje homogenog dela

$x(t) = \sum_{k=1}^{n}(\phi_k(t) + C_k)X_k(t) = \sum_{k=1}^{n} C_kX_k(t) + \sum_{k=1}^{n}\phi_k(t)X_k(t)$ $=X_h(t) + X^*(t)$

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8. <u>Integracija linearnog sistema sa konstantnim koeficijentima</u>

$\frac{dX_i}{dt} = a_{ij}x_j(t) + f_i(t) \ \ (i = 1, \dots, n\,; t \in (\alpha, \beta); a_{ij} = const.$

Za slucaj **n=3**

$\frac{dx}{dt} = a_1x + b_1y + c_1z + f_1$

$\frac{dy}{dt} = a_2x + b_2y + c_2z + f_2$

$\frac{dz}{dt} = a_3x + b_3y + c_3z + f_3$

*Homogeni sistem:

$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j(t)$ i=1,...,n

$$\frac{dX}{dt} = A\cdot X \ i\ za\ slucaj\ \boldsymbol{n = 3} \quad A = \begin{bmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{bmatrix}$$

X(t)= $A_\alpha \cdot e^{\lambda t}$ $\quad \lambda \in \boldsymbol{R}$

resenje trazimo u obliku $\begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{bmatrix} e^{\lambda t}$

za n=3 vazi da je x= $\alpha e^{\lambda t}$, y=β$e^{\lambda t}$, z=γ$e^{\lambda t}$ => (A-λI) · A_α=0 matrica

za n=3 vazi $\begin{bmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{bmatrix}$=k($\lambda$)

Koreni karakteristicne jednacine nazivaju se sopstvene vrednosti matrice A. Od prirode resenja po λ karak. jedn. (da li su realni ili kompleksni) i njihove visestrukkosti zavisi oblik resenja sistema.

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9. Parcijalne jednacine. Formiranje i tipovi resenja

u=u(x,y) u=u(x,y,z)

$\frac{\partial u}{\partial x} = xy + 1 \quad u(x,y) = ?$

$\int \frac{\partial u}{\partial x} dx = \int (xy+1)dx + \varphi(y) \;\; => \;\; u(x,y) = \frac{x^2 y}{2} + x + \varphi(y)$

$\frac{\partial^2 u}{\partial x \partial y} = 4 \; pa \; je \; po \; x \;\; \frac{\partial y}{\partial y} = 4x + \varphi(y) \; pa \; je \; po \; y \;\; u(x,y) = 4xy + \int \varphi(y)dy + \Psi(x)$

Def. Neka je $u(x_1, x_2, \ldots, x_n)$ funkcija n promenljivih i ima parcijalne izvode do reda m, gde je m=1,...,n. Tada je:

$F\left(x_1, x_2, \ldots, x_n, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^m u}{\partial x^m}\right) = 0$ tada je parc. jednacina m-tog reda. Posebno:

$F\left(x_1, x_2, \ldots, x_n, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$ je parcijalne jednacina prvog reda.

n=2 $F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$

n=3 $F\left(x, y, z, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0$

Def. Bilo koja funkcija $x_1, x_2, \ldots, x_n \to u(x_1, x_2, \ldots, x_n)$ koja zajedno sa svojim parcijalnim izvodima zadovoljava jednnacinu * naziva se resenje parcijalne diferencijalne jednacine.

Formiranje jednacina:

$u(x,y) = f(x^2+y^2) \quad \frac{\partial u}{\partial x} = 2x \cdot f' \quad \frac{\partial u}{\partial y} = 2y \cdot f' \quad \frac{1}{2x} \cdot \frac{\partial u}{\partial x} = \frac{1}{2y} \cdot \frac{\partial u}{\partial y}$

$$\boldsymbol{y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0}$$

Vrste resenja: $F(x, y, u, u_x, u_y) = 0$ (1)

Def1: Ono resenje jednacine (1) koje sadrzi dve proizvoljne konstante cijom se eliminacijom dolazi iskljucivo do date jednacine naziva se potpuno resenje (potpuni integral).

Def2: Resenje jednacine (1) koje sadrzi proizvoljnu funkciju cijom se eliminacijom dolazi iskljucivo do date jednacine naziva se opste resenje ili opsti integral.

$F(x, y, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0$ (2)

Def3: Potpuni integral parcijalne jednacine drugog reda (2) je ono resenje koje sadrzi 5 proizvoljnih konstanti cijom se eliminacijom dolazi do date jednacine.

Def4: Opste resenje parcijalne jednacine drugog reda (2) je ono njeno resenje koje sadrzi dve promenljive funkcije cijom se eliminacijom dobija data jednacina.

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10. Liinearna homogena parcijalna jednacina prvog reda

$X_1(x_1, x_2, \dots, x_n)\frac{\partial u}{\partial x_1} + X_2(x_1, x_2, \dots, x_n)\frac{\partial u}{\partial x_2} + \cdots + X_n(x_1, x_2, \dots, x_n)\frac{\partial u}{\partial x_n} = 0$ **(H)**

za n=2 $P(x, y)\frac{\partial u}{\partial x} + Q(x, y)\frac{\partial u}{\partial y} = 0$

za n=3 $P(x, y, z)\frac{\partial u}{\partial x} + Q(x, y, z)\frac{\partial u}{\partial y} + R(x, y, z)\frac{\partial u}{\partial z} = 0$ gde su X_i date neprekidne funkcije.

$$\frac{dx_1}{X_1(x_1, x_2, \dots, x_n)} = \frac{dx_2}{X_2(x_1, x_2, \dots, x_n)} = \cdots = \frac{dx_n}{X_n(x_1, x_2, \dots, x_n)} \qquad \textbf{(S)}$$

za n=2 $\frac{dx}{P} = \frac{dy}{Q}$

za n=3 $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Neka je $F(x_1, x_2, \dots, x_n) = C$ jedno resenje sistema (S). $F(x_1, x_2, \dots, x_n) = C \Rightarrow dF = 0$ ekvival. sa

$\frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + \cdots + \frac{\partial F}{\partial x_n}dx_n = 0$

Iz S sledi $dx_1 = \lambda \cdot X_1, dx_2 = \lambda \cdot X_2, \dots, dx_n = \lambda \cdot X_n$

$\lambda\left(X_1\frac{\partial F}{\partial x_1}+\cdots+X_n\frac{\partial F}{\partial x_n}\right)=0 => jer\ je\ \lambda\neq 0$

da je $X_1\frac{\partial F}{\partial x_1}+\cdots+X_n\frac{\partial F}{\partial x_n}=0$ a ovo znaci da je f-ja F resenje jednacine (H) i obrnuto.

Neka je $u=f(x_1,\dots,x_n)$ jedno resenje jednacine (H) => $X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=0$.

Iz (S) imamo da je $X_1=\frac{\partial x_1}{\lambda},\dots,X_n=\frac{\partial x_n}{\lambda}$ pa zamenom u $X_1\frac{\partial f}{\partial x_1}$ je $\frac{1}{\lambda}\left(\frac{\partial f}{\partial x_1}dx_1+\cdots+\frac{\partial f}{\partial x_n}dx_n\right)=0$

$\frac{\partial f}{\partial x_1}dx_1+\cdots+\frac{\partial f}{\partial x_n}dx_n => df=0 \leftrightarrow \boldsymbol{f=C}$

<u>T:</u> Integracija jednacine (H) i integracija sistema (S) su ekvivalentni problemi. Sistem (S) naziva se *sistem karakteristika za jednacinu (H).*

<u>T:</u> Neka su $f_1(x_1,\dots,x_n),f_2(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n)$ resenja jednacine (H) i neka je ϕ(...) proizvoljna diferencijabilna funkcija. Tada je $u(x_1,\dots,x_n)=\phi(f_1(\dots),f_2(\dots),\dots,f_{n-1}(\dots))$ opste resenje jedn. (H)

Specijalno ako imamo poc. uslove (Kosijeve uslove) tada se proizvoljna f-ja ϕ moze odrediti i tada imamo Kosijev problem, tj. Kosijevo resenje.

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11. <u>Nehomogena kvazilinearna parc. jedn. 1. reda</u>

$$X_1(x_1,x_2,\dots,x_n)\frac{\partial u}{\partial x_1}+X_2(x_1,x_2,\dots,x_n)\frac{\partial u}{\partial x_2}+\cdots+X_n(x_1,x_2,\dots,x_n)\frac{\partial u}{\partial x_n}=X_{n+1}(x_1,x_2,\dots,x_n,u)$$

Ova jednacina se svodi na homogenu.

Neka je $\phi(x_1,x_2,\dots,x_n,u)=C$ jedno resenje date jednacine. Tada je:

$$\frac{\partial\phi}{\partial x_1}+\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x_1}=0,\frac{\partial\phi}{\partial x_2}+\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x_2}=0,\dots,\frac{\partial\phi}{\partial x_n}+\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x_n}=0$$

$$\frac{\partial u}{\partial x_1}=-\frac{1}{\frac{\partial\phi}{\partial u}}\cdot\frac{\partial\phi}{\partial x_1},\dots,\ \frac{\partial u}{\partial x_n}=-\frac{1}{\frac{\partial\phi}{\partial u}}\cdot\frac{\partial\phi}{\partial x_n}$$

$-\frac{1}{\frac{\partial\phi}{\partial u}}\cdot\left(X_1\frac{\partial\phi}{\partial x_1}+\cdots+X_n\frac{\partial\phi}{\partial x_n}\right)=X_{n+1}$ i sve to mnozimo sa - $\frac{\partial\phi}{\partial u}$

$$X_1\cdot\frac{\partial\phi}{\partial x_1}+X_2\cdot\frac{\partial\phi}{\partial x_2}+\cdots+X_n\cdot\frac{\partial\phi}{\partial x_n}+X_{n+1}\cdot\frac{\partial\phi}{\partial u}=0$$

ϕ je resenje ove homogene jednacine ciji je sistem karakteristika:

$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}=\cdots=\frac{dx_n}{X_n}=\frac{du}{X_{n+1}}$ --> n prvih integrala mozemo naci odavde. o.r. F($f_1,f_2,\dots,f_n$)=0